Semiclassical stochastic representation of the Feynman integral

1994 J. Phys. A: Math. Gen. 27 6457

(http://iopscience.iop.org/0305-4470/27/19/018)

J. Phys. A: Math. Gen. 27 (1994) 6457–6477. Printed in the UK

Semiclassical stochastic representation of the Feynman integral

Z Haba

Institute of Theoretical Physics, University of Wrocław, Wrocław, Poland

Received 21 April 1994, in final form 1 July 1994

Abstract. We show that for a wide class of analytic potentials the solution of the Schrödinger equation can be expressed by the Wiener integral. We express the Feynman kernel by the Brownian bridge. We prove an asymptotic semiclassical expansion of the solution of the Schrödinger equation as well as of the Feynman kernel. We show that for a small time the van Vleck formula holds true as the limit $\hbar \rightarrow 0$ of the Feynman kernel.

1. Introduction

There have been numerous approaches to the mathematical theory of the Feynman path integral [1]. An approach suggested by Gelfand and Yaglom [2] to use the propagation kernel in order to construct a complex measure appeared to be wrong [3]. Another approach based on a Fourier transform suggested first by Ito [4] and developed by Albeverio and Hoegh-Krohn [5] defines a complex measure (Fresnel integral), which, however, is not supported by paths in the configuration space. A relation of such a measure to summation over polygonal paths is discussed in Elworthy and Truman [6]. There have been numerous works concerned with an analytic continuation of the Wiener integral to the Feynman integral; let us mention Cameron [3], Ito [7], Nelson [8] (see also [9]).

The difficulty with a naive definition of the Feynman integral originates from the non-existence of the Lebesgue measure in an infinite number of dimensions. There are many reasons for using the Wiener measure in an infinite number of dimensions to fulfil the role of the Lebesgue measure. An interpretation of the Feynman integral as a generalized Wiener functional (i.e. a distribution) in an infinite number of dimensions has been discussed in [10]. The class of functionals which are integrable in various approaches is quite restricted. If we confine ourselves to analytic wavefunctions and severely restrict the class of potentials which are admitted, then the Feynman integral can be expressed by the Wiener integral according to Cameron [3]. This expression has a rigourous mathematical meaning for a class of potentials discussed by Doss [11] and Azencott and Doss [12], who developed some ideas of Cameron [3].

In this paper we further develop the approach of Cameron–Doss–Azencott. We extend the class of potentials which can be treated this way to Doss potentials [11] plus a Fourier transform of a bounded measure. We first regularize the potential and then remove the regularization. In this case the Feynman formula should be understood in a distributional sense, because the Feynman integral is a limit of Wiener integrals with regularized potentials. We show some analyticity properties of the resulting wavefunctions. Next, we remove the requirement on the wavefunction to be analytic (sections 3 and 4). We derive a formula for the Feynman kernel in terms of the Brownian bridge. The Feynman kernel can be used to extend the probabilistic representation of the solution of the Schrödinger equation to non-analytic wavefunctions. In sections 5 and 6 we apply the probabilistic representation of the Feynman integral to the semiclassical expansion. We show that standard (but formal) functional integral methods can be given a rigourous meaning in such a way that the whole intuitive appeal of the Feynman integral is preserved. We show that the semiclassical expansion is asymptotic for sufficiently small time.

2. Solution of the Schrödinger equation

A representation of the solution of the Schrödinger equation in terms of Brownian motion has already been discussed by Cameron [3]. This approach to the Feynman integral has been further developed by Doss [11] and Azencott and Doss [12]. We shall show in this section that the representation in terms of the Wiener measure can be applied to the class of potentials considered by Ito [4] and Albeverio and Hoegh-Krohn [5].

Let $\mathcal{F}_n$ denote the Banach algebra of functions of the form

$$f(x) = \int \exp(\mathrm{i}\gamma x)\,\mathrm{d}\nu_f(\gamma) \tag{2.1}$$

where ν is a complex measure on $\mathbb{R}^n$ with a bounded variation $|\nu|$ (this is the class of functions discussed in [4–5]). We have to restrict this class further to analytic functions (such a restriction has also been introduced in [13]). We need yet another restriction, which comes out in one step of the proof of theorem 2.2 below. So we consider the set $\mathcal{H}_n{}^{\mathcal{F}} \subset \mathcal{F}_n$ defined by

$$\{\forall_{\epsilon>0}\ \exists_K \mid \forall_{a\in\mathbb{R}} \int \exp(a|\gamma|)\,\mathrm{d}|\nu|(\gamma) \leqslant K(\epsilon)\exp(\epsilon a^2)\}. \tag{2.2}$$

Both $\mathcal{F}_n$ and $\mathcal{H}_n{}^{\mathcal{F}}$ are dense subsets in $L_2(\mathbb{R}^n)$.

We are interested in a path integral representation of the solution $\psi_t \equiv U_t\psi$ of the Schrödinger equation

$$\mathrm{i}\hbar\partial_t\psi_t(x) = \left(-\frac{\hbar^2}{2m}\Delta + gV(x)\right)\psi_t(x) \equiv (H\psi_t)(x) \qquad \psi_t|_{t=0} = \psi \tag{2.3}$$

where $\psi, V \in \mathcal{H}_n{}^{\mathcal{F}}$. We shall also consider a complex extension of equation (2.3) in the form

$$\partial_t\psi_t(x) = \left(\tfrac{1}{2}\sigma^2\lambda^2\Delta + \frac{g}{\hbar\lambda^2}V(x)\right)\psi_t(x) \tag{2.4}$$

where $\lambda \in \mathbb{C}$ and

$$\sigma = \sqrt{\frac{\hbar}{m}}$$

(clearly equation (2.3) corresponds to $\lambda = \sqrt{\mathrm{i}} = (1+\mathrm{i})/\sqrt{2}$).

We express the path integral by the Wiener process b_τ (Brownian motion). $b_\tau \in \mathbb{R}^n$ is the Gaussian process with the covariance (when there is a danger of confusing vector indices with time we shall also use the notation $b(\tau)$ instead of b_τ)

$$E[b^k(\tau)b^r(s)] = \delta_{kr}\min(\tau, s) \qquad k, r = 1, 2, \ldots, n. \tag{2.5}$$

We use the notation

$$E[F] \equiv \int \mathrm{d}\mu_{W}{}^{t}(b)\, F(b)$$

where $\mu_{W}{}^{t}$ is the Wiener measure and the integral is over the Wiener space $C([0,t])$ which consists of continuous functions defined on the interval $[0,t]$ with $b(0)=0$.

We need first the following.

Lemma 2.1. Let μ be a Gaussian measure on the Wiener space $C([0,t])$. Assume that Q is a non-negative quadratic form on $C([0,t])$ such that $Q = \lim_{p\to\infty} Q_p$ with probability 1, where Q_p is a continuous non-negative quadratic form. Let $\mathcal{L}(b) = \sum_{k=1}^{r} a_k b(\tau_k)$, where $a_k \in \mathbb{R}$ $(k=1,2,\ldots,r)$ and $0 \leqslant \tau_k \leqslant t$ are arbitrary. Then

$$\left|\int \mathrm{d}\mu(b) \exp(-Q(b) - \mathrm{i}\lambda\mathcal{L}(b))\right| \leqslant 1 \tag{2.6}$$

if $\mathrm{Re}(\lambda^2) \geqslant 0$.

Proof. For Q_p we have explicitly (see [14], theorem 4, section 18)

$$\int \mathrm{d}\mu(b) \exp(-Q(b) - \mathrm{i}\lambda\mathcal{L}(b))$$

$$= \det(1 + G^{1/2} Q_p G^{1/2})^{-1/2} \exp\left(-\tfrac{1}{2}\lambda^2 \sum_{s,q=1}^{r} a_s a_q C_p(\tau_s, \tau_q)\right)$$

where

$$C_p = G^{1/2}(1 + G^{1/2} Q_p G^{1/2})^{-1} G^{1/2}$$

and G is the covariance of μ. It follows that the bound (2.6) holds true for Q_p. Then, $\exp(-Q_p - \mathrm{i}\lambda\mathcal{L})$ is bounded by an integrable function $\exp|\lambda\mathcal{L}|$. Hence, equation (2.6) holds true by Lebesgue dominated convergence. □

With a general potential of the form (2.1) we need a cutoff on an intermediate stage

$$\theta_R(x) = \exp\left(-\frac{|x|^2}{2R}\right).$$

Let us denote

$$\Omega_t(g\theta_R(b); V(x + \lambda\sigma b)) = \exp\left\{\frac{g}{\hbar\lambda^2}\int_0^t \theta_R(b_\tau) V(x + \lambda\sigma b_\tau)\,\mathrm{d}\tau\right\}. \tag{2.7}$$

We shall also use a shorthand notation $\Omega(g\theta_R)$ for the LHS of equation (2.7).

Theorem 2.2. Assume that $\psi, V \in \mathcal{H}_n{}^{\mathcal{F}}$. Then, $E[\Omega_t(g\theta_R)]$ is an analytic function of $g \in \mathbb{C}$ and $\lambda \neq 0$ as long as $\mathrm{Re}\,\lambda^2 \geqslant 0$. Define

$$\psi_t{}^R(x) = E[\Omega_t(g\theta_R)\theta_R(b_t)\psi(x + \lambda\sigma b_t)]. \tag{2.8}$$

Then, $\psi_t(x) \equiv \lim_{R\to\infty} \psi_t{}^R(x)$ exists uniformly in t and x. $\psi_t(x)$ solves the Schrödinger equation (2.3). The solution is an analytic function of g and λ (in the same region where $E[\Omega_t]$ is analytic).

Proof. The functional under the expectation value in equation (2.8) is bounded in b. In fact, the exponent in equation (2.7) is bounded because

$$|\theta_R(b_\tau)V(x+\lambda\sigma b_\tau)| = \left|\int \mathrm{d}\nu_V(\gamma)\exp\left(-\frac{b_\tau^2}{2R}\right)\exp(\mathrm{i}\gamma(x+\lambda\sigma b_\tau))\right|$$

$$\leqslant \int \mathrm{d}|\nu_V|(\gamma)\exp\left(|\gamma|\,|\lambda\sigma|\,|b_\tau| - \frac{b_\tau^2}{2R}\right) \leqslant K(R) \tag{2.9}$$

owing to the assumption (2.2).

Hence, Ω as well as $\theta\psi$, is bounded. We expand Ω in a power series in g. Then, applying the Lebesgue dominated convergence theorem we exchange the sum with the expectation value. We obtain

$$\psi_t{}^R = \lim_{N\to\infty} F_N(R) \tag{2.10}$$

where

$$F_N(R) = \sum_{n=0}^{N}\left(\frac{g}{\lambda^2\hbar}\right)^n \frac{1}{n!}$$

$$\times E\left[\int_0^t \mathrm{d}\tau_1 \ldots \int_0^t \mathrm{d}\tau_n \mathrm{d}\nu_V(\gamma_1)\ldots\mathrm{d}\nu_V(\gamma_n)\,\mathrm{d}\nu_\psi(\beta)\right.$$

$$\left.\times \exp\left(-\mathcal{B}_n{}^R + \mathrm{i}\left(\sum_{k=1}^{n}\gamma_k + \beta\right)x\right)\right]$$

with

$$\mathcal{B}_n{}^R = \frac{1}{2R}\left(b_t{}^2 + \sum_{k=1}^{n} b_{\tau_k}{}^2\right) - \mathrm{i}\sigma\lambda\left(\beta b_t + \sum_{k=1}^{n}\gamma_k b_{\tau_k}\right).$$

Then, owing to the assumption (2.2) we can apply the Fubini theorem

$$E\left[\int \mathrm{d}\nu_V(\gamma_1)\ldots\mathrm{d}\nu_V(\gamma_n)\mathrm{d}\nu_V(\beta)\exp(-\mathcal{B}_n{}^R)\right]$$

$$= \int \mathrm{d}\nu_V(\gamma_1)\ldots\mathrm{d}\nu_V(\gamma_n)\mathrm{d}\nu_\psi(\gamma_n)\mathrm{d}\nu_\psi(\beta)\,E[\exp(-\mathcal{B}_n{}^R)].$$

We are now able to show the existence of the limit

$$\lim_{R\to\infty} F_N(R) \equiv F_N. \tag{2.11}$$

This follows from the Lebesgue dominated convergence theorem, because the measure $|\nu|$ is finite and

$$|E[\exp(-\mathcal{B}_n{}^R)]| \leqslant 1 \tag{2.12}$$

by virtue of lemma 2.1. Note that from the bound (2.12) it follows that

$$|F_N(R) - F_M(R)| \leqslant \sum_{n=M}^{N} \left(\frac{|g|t}{\hbar|\lambda|^2}\right)^n \frac{|\nu|^n}{n!}$$

where $|\nu|$ is the total variation of ν.

Hence, the convergence $N \to \infty$ of $F_N(R)$ is uniform in R (as well as in $\boldsymbol{x}$ and t). Therefore we may exchange the limits $N \to \infty$ and $R \to \infty$

$$\begin{aligned}\psi_t &\equiv \lim_{R\to\infty} \psi_t{}^R = \lim_{R\to\infty}\lim_{N\to\infty} F_N(R) = \lim_{N\to\infty}\lim_{R\to\infty} F_N(R) \qquad (2.13)\\ &= \sum_{n}^{\infty} (\lambda^{-2})^n \hbar^{-n} \frac{g^n}{n!} E\left[\left(\int V(\boldsymbol{x} + \lambda\sigma \boldsymbol{b}_\tau)\,\mathrm{d}\tau\right)^n \psi(\boldsymbol{x} + \lambda\sigma\boldsymbol{b}_t)\right]\\ &= \sum_{n}^{\infty} (\lambda^{-2})^n \hbar^{-n} \frac{g^n}{n!} \int_0^t \mathrm{d}\tau_1 \ldots \int_0^t \mathrm{d}\tau_n \int \mathrm{d}\nu_V(\gamma_1)\ldots\mathrm{d}\nu_V(\gamma_n)\,\mathrm{d}\nu_\psi(\beta)\\ &\quad\times \exp\left\{\left(\mathrm{i}\sum_k \gamma_k + \mathrm{i}\beta\right)x\right\} \exp\left\{-\tfrac{1}{2}\sigma^2\lambda^2 E\left[\left(\sum_k \gamma_k b_{\tau_k} + \beta b_t\right)^2\right]\right\}.\end{aligned}$$

It is clear from equation (2.13) that if $|\nu_V|$ and $|\nu_\psi|$ are bounded, $\lambda \neq 0$ and $\operatorname{Re}\lambda^2 \geqslant 0$, then the series (2.13) is convergent uniformly in t and $\boldsymbol{x}$. It can also be checked by direct calculation that ψ_t (2.13) satisfies the Schrödinger equation. □

We can obtain an immediate extension of the formula (2.8) to potentials of the form

$$V^Q{}_t(\boldsymbol{x}) = \tfrac{1}{2}m\omega(t)^2\boldsymbol{x}^2 + c(t)\boldsymbol{x} + gV_t(\boldsymbol{x}) \equiv Q_t(\boldsymbol{x}) + gV_t(\boldsymbol{x}) \qquad (2.14)$$

where $V \in \mathcal{H}_n{}^{\mathcal{F}}$ and ω and c are continuous functions of time.

Lemma 2.3. Assume that $\omega(t)^2 \leqslant \omega_0^2$ for each t, then

$$E[\exp Q] \equiv E\left[\exp\left\{\frac{1}{\lambda^2\hbar}\int_0^t Q_\tau(\boldsymbol{x} + \lambda\sigma\boldsymbol{b}_\tau)\,\mathrm{d}\tau\right\}\right] < \infty \qquad (2.15)$$

if $t < \pi/\omega_0$.

Proof. The quadratic part in the exponential is equal to $\frac{1}{2}\omega(t)^2 b_\tau^2 \leqslant \frac{1}{2}\omega_0^2 b_\tau^2$. We can bound the expectation value in equation (2.15) by the expectation value of an exponential of a linear function in b (which is know to be finite) and the expectation value of the exponential of the quadratic part bounded by

$$E\left[\exp\left\{\tfrac{1}{2}\int_0^t \omega_0^2 b_\tau^2\,\mathrm{d}\tau\right\}\right] = \int \prod_{n=1}^{\infty} \frac{\mathrm{d}b_n}{\sqrt{2\pi}}\, n \exp\left(-\tfrac{1}{2}n^2 b_n{}^2 + \frac{1}{2\pi^2}\,\omega_0^2 t^2 b_n{}^2\right) \qquad (2.16)$$

where an expansion of the Brownian motion in an orthonormal basis of $L_2(0,t)$ has been applied (see [16]). The statement of the lemma follows from finiteness of the integral (2.16). □

Theorem 2.4. If $t < \pi/\omega_0$ then theorem 2.2 holds true for V^Q (equation (2.14)).

Proof. We define a new measure by the formula

$$E_Q[F] \equiv E[\exp Q\ F]. \tag{2.17}$$

Equation (2.17) determines a finite measure as long as $t < \pi/\omega_0$. Then lemma 2.1, as well as the arguments used in the proof of theorem 2.2, remain true with the replacement $E[\cdot] \to E_Q[\cdot]$, because in the proof we have used only the fact that the measure is Gaussian and not its explicit form. □

As a further extension of the formula (2.8) we consider the Schrödinger equation in an electromagnetic field

$$\partial_t \psi_t(x) = \left(\frac{\hbar\lambda^2}{2m}\nabla_A^2 + \frac{g}{\hbar\lambda^2}V_t\right)\psi_t(x) \tag{2.18}$$

where

$$(\nabla_A)_k = \partial_k + \mathrm{i}eA_k\,.$$

Denote

$$\begin{aligned}\Omega_t{}^A(g\theta_R) &\equiv S_R{}^A\Omega(g\theta_R) \\ &= \exp\left\{\frac{\mathrm{i}}{\hbar}\lambda\sigma e\int_0^t \theta_R(b_\tau)A(x+\lambda\sigma b_\tau)\circ \mathrm{d}b_\tau\right\} \\ &\quad\times \exp\left\{\frac{g}{\hbar\lambda^2}\int_0^t \theta_R(b_\tau)V(x+\lambda\sigma b_\tau)\,\mathrm{d}\tau\right\}.\end{aligned} \tag{2.19}$$

Here $\theta_R A$ is a bounded function. The circle denotes the Stratonovich stochastic integral (see [17]). It is related to the Ito integral (without the circle) used later on (and in equation (2.20) below) by

$$\int A\circ \mathrm{d}b_\tau = \int A\,\mathrm{d}b_\tau + \tfrac{1}{2}\int \operatorname{div} A\,\mathrm{d}\tau.$$

We need first the estimate of Carlen and Kree [15] (proposition 3).

Lemma 2.5. Let $f_\tau(x)$ be a bounded function of τ and x. Then

$$E\left[\exp\left(\rho\int_0^t f_\tau(b_\tau)\,\mathrm{d}b_\tau\right)\right] < \infty \tag{2.20}$$

is an analytic function of ρ.

Theorem 2.6. Assume $A, V, \psi \in \mathcal{H}_n{}^{\mathcal{F}}$. Define

$$\psi_t{}^R(x) = E[\Omega_t{}^A(g\theta_R)\psi(x+\lambda\sigma b_t)]. \tag{2.21}$$

Then $\lim_{R\to\infty}\psi_t{}^R(x) \equiv \psi_t(x)$ exists uniformly in t and x. ψ_t is the unique solution of the Schrödinger equation (2.18) with the initial condition ψ. ψ_t is an analytic function of $g \in \mathbb{C}$, $e \in \mathbb{C}$ and $\lambda \neq 0$ as long as $\operatorname{Re}\lambda^2 \geqslant 0$. $\psi_t(x)$ has an analytic continuation $\psi_t(x+\mathrm{i}y)$ to $\mathbb{C}^n$.

Proof. From equation (2.19) $\Omega_t{}^A(g\theta_R) = S_R{}^A\Omega_t(g\theta_R)$, where the first factor is bounded by an integrable function as a result of lemma 2.5, and the second factor is bounded owing to the estimate (2.9). We expand $\Omega_t{}^A(g\theta_R)$ in a power series in g and e. As in the proof of theorem 2.2, applying the Lebesgue dominated convergence theorem, we exchange the sum with the expectation value. We obtain an analogue of the series $F_N(R)$ in equation (2.10). Then, using the estimates of Carlen and Kree [15] on $E[(\int f\,\mathrm{d}b)^k]$ (implicitly contained in equation (2.20)) we can show that the convergence $N \to \infty$ of $F_N(R)$ is uniform in R. Hence we can exchange the limits $N \to \infty$ and $R \to \infty$ (as we did in equation (2.13)) proving that $\psi_t(x)$ is a sum of an absolutely convergent series in e and g. □

3. Stochastic representation of the Feynman propagator

In section 2 we have restricted ourselves to wavefunctions which are boundary values of holomorphic functions $\mathcal{H}_n{}^{\mathcal{F}}$. This is a dense set in $L_2(\mathbb{R}^n)$. So a definition of U_t on this set determines U_t in a unique way. However, we cannot extend U_t to $L_2(\mathbb{R}^n)$ directly through the formula (2.8). We begin with a definition of a semigroup S_t and its kernel on $L_2(\mathbb{R}^{2n})$. Then, through a restriction of the kernel to analytic functions we are able to define the Feynman propagator $U_t(x, y)$ for Doss potentials [11] as well as the potentials of the form of a Fourier transform of a bounded measure.

Let us consider a decomposition of $\mathbb{R}^{2n}$ into $\mathbb{R}^n \oplus \mathbb{R}^n$, $z = (x_1, x_2)$, where $x_1, x_2 \in \mathbb{R}^n$. We define a subset $\mathbb{G}_z$ of $\mathbb{R}^{2n}$

$$\mathbb{G}_z \subset \mathbb{R}^{2n} \equiv \{w \in \mathbb{R}^{2n} | w = (x_1 + y, x_2 + y), y \in \mathbb{R}^n\} \tag{3.1}$$

Let $\mathcal{V}$ be a complex-valued potential defined on $\mathbb{R}^{2n}$ such that on $\mathbb{G}_z$

$$|\operatorname{Im} \mathcal{V}(x_1 + y, x_2 + y)| \leqslant A(x_1, x_2)(|y| + 1) \tag{3.2}$$

with a certain continuous non-negative function A. We define a semigroup S_t on the set of functions defined on $\mathbb{R}^{2n}$ such that on $\mathbb{G}_z$

$$|\Phi(x_1 + y, x_2 + y)| \leqslant M(x_1, x_2) \exp(C(x_1, x_2)|y|) \tag{3.3}$$

with certain non-negative continuous functions C and M. S_t is defined by the Feynman–Kac formula

$$\begin{aligned}(S_t\Phi)(z) &= E\left[\exp\left(-\frac{\mathrm{i}}{\hbar}\int_0^t \mathcal{V}\left(x_1 + \frac{\sigma}{\sqrt{2}} b_\tau, x_2 + \frac{\sigma}{\sqrt{2}} b_\tau\right) \mathrm{d}\tau\right)\right.\\ &\qquad \left.\times\, \Phi\left(x_1 + \frac{\sigma}{\sqrt{2}} b_t, x_2 + \frac{\sigma}{\sqrt{2}} b_t\right)\right]\\ &= \int E\left[\delta\left(y_1 - x_1 - \frac{\sigma}{\sqrt{2}} b_t\right)\delta\left(y_2 - x_2 - \frac{\sigma}{\sqrt{2}} b_t\right)\right.\\ &\qquad \left.\times \exp\left(-\frac{\mathrm{i}}{\hbar}\int_0^t \mathcal{V}\left(x_1 + \frac{\sigma}{\sqrt{2}} b_\tau, x_2 + \frac{\sigma}{\sqrt{2}} b_\tau\right)\mathrm{d}\tau\right)\right] \Phi(y_1, y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2\\ &\equiv \int S_t(x_1, x_2; y_1, y_2)\Phi(y_1, y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2\end{aligned} \tag{3.4}$$

where

$$\sigma = \sqrt{\frac{\hbar}{m}}\,.$$

The last two lines of equation (3.4) define the kernel of the operator S_t. It can be seen from equation (3.4) that under the expectation value the argument of the functions $\mathcal{V}$ and Φ runs over the subset $\mathbb{G}_z \subset \mathbb{R}^{2n}$. Under the assumptions (3.2)–(3.3) the integrand (3.4) is bounded on $\mathbb{G}_z$ by

$$\exp\left(K(x_1, x_2)\left\{\sup_{0\leqslant\tau\leqslant t} |b_\tau|\right\}\right)$$

Hence the expectation value is finite.

We can express the kernel $S_t(x_1, x_2; y_1, y_2)$ of S_t by the Brownian bridge α. The Brownian bridge (see [16]) is the Gaussian process defined on the interval [0, 1] with the covariance

$$E[\alpha_k(s)\alpha_r(s')] = \delta_{kr}s(1-s') \qquad \text{if} \qquad s \leqslant s' \tag{3.5}$$

and the boundary conditions $\alpha(0) = \alpha(1) = 0$. The δ-functions in equation (3.4) depend only on the final time t. Then the expectation value in the second line of equation (3.4) is equal to the expectation value

$$E\left[\delta\left(y_1 - x_1 - \frac{\sigma}{\sqrt{2}}b_t\right)\delta\left(y_2 - x_2 - \frac{\sigma}{\sqrt{2}}b_t\right)\right]$$
$$= (4\pi t\sigma^2)^{-n/2}\exp\left(-\frac{(x_1 + x_2 - y_1 - y_2)^2}{4t\sigma^2}\right)\delta(x_1 - x_2 - y_1 + y_2)$$

times the expectation value of $\exp(-(\mathrm{i}/\hbar)\int\mathcal{V})$, where the Brownian motion is constrained to end at y. A solution of this constraint gives a simple formula for the kernel (a similar formula has been derived in [16])

$$S_t(x_1, x_2; y_1, y_2) = (4\pi t\sigma^2)^{-n/2}\exp\left(-\frac{(x_1 + x_2 - y_1 - y_2)^2}{4t\sigma^2}\right)\delta(x_1 - x_2 - y_1 + y_2)$$
$$\times E\left[\exp\left\{-\frac{\mathrm{i}}{\hbar}\int_0^t \mathcal{V}\left(x_1\left(1-\frac{s}{t}\right) + y_1\frac{s}{t} + \sqrt{\tfrac{1}{2}t}\,\sigma\alpha\left(\frac{s}{t}\right),\right.\right.\right.$$
$$\left.\left.\left. x_2\left(1-\frac{s}{t}\right) + y_2\frac{s}{t} + \sqrt{\tfrac{1}{2}t}\,\sigma\alpha\left(\frac{s}{t}\right)\right)\mathrm{d}s\right\}\right]. \tag{3.6}$$

We wish to relate the kernel (3.6) to the Feynman kernel of the unitary evolution (2.3). This can be achieved if we restrict oursevles to functions of the form

$$\mathcal{V}(x_1, x_2) \equiv V(x_1 + \mathrm{i}x_2) \qquad \Phi(x_1, x_2) \equiv \phi(x_1 + \mathrm{i}x_2) \tag{3.7}$$

where V and ϕ are holomorphic functions of their arguments. Inserting equation (3.7) into equation (3.4) we obtain

$$(S_t\phi)(x_1 + \mathrm{i}x_2) = (4\pi t\sigma^2)^{-n/2}\int \mathrm{d}y\,\phi(\tfrac{1}{2}(y + x_1 + x_2) + \mathrm{i}\tfrac{1}{2}(y - x_1 + x_2))$$
$$\times \exp\left(-\frac{(y - x_1 - x_2)^2}{4t\sigma^2}\right)\mathcal{E}_t(y, x_1, x_2) \tag{3.8}$$

where by $\mathcal{E}$ we have denoted $E[\dots]$ (the last factor in equation (3.6)) with $y_1 = \frac{1}{2}(y + x_1 - x_2)$ and $y_2 = \frac{1}{2}(y - x_1 + x_2)$.

We show next that

$$(4\pi t\sigma^2)^{-n/2} \int dy\, \chi\big(\tfrac{1}{2}(y+x) + \mathrm{i}\tfrac{1}{2}(y-x)\big) \exp\left(-\frac{(x-y)^2}{4t\sigma^2}\right)$$

$$= (2\pi \mathrm{i} t\sigma^2)^{-n/2} \int dy\, \chi(y) \exp\left(-\frac{(x-y)^2}{2\mathrm{i}t\sigma^2}\right) \tag{3.9}$$

(where $\mathrm{i}^{n/2} = \exp(\frac{1}{2}\mathrm{i}n\pi)$) for any analytic function $\chi \in L_1(\mathbb{R}^n)$ of the form

$$\chi(y) = \int dp\, \tilde{\chi}(p) \exp \mathrm{i}py\,. \tag{3.10}$$

In order to check equation (3.9) we insert equation (3.10) into equation (3.9), apply the Fubini theorem, and perform the y-integral.

Finally, we set $x_2 = 0$ in equation (3.8) and apply the identity (3.9). We obtain an expression for $(S_t\phi)(x)$ in terms of the kernel

$$K_t(x,y) = (2\pi \mathrm{i} t\sigma^2)^{-n/2} \exp\left(-\frac{(x-y)^2}{2\mathrm{i}t\sigma^2}\right)$$

$$\times E\left[\exp\left\{-\frac{\mathrm{i}}{\hbar}\int_0^t V\left(x\left(1-\frac{s}{t}\right) + y\frac{s}{t} + \lambda\sqrt{t}\,\sigma\alpha\left(\frac{s}{t}\right)\right) ds\right\}\right]$$

$$\equiv K_t{}^0(x,y)\mathcal{R}_t(x,y) \tag{3.11}$$

where

$$\lambda = \sqrt{\mathrm{i}} \equiv \frac{1}{\sqrt{2}}(1+\mathrm{i})$$

and we have denoted the free propagator by K^0.

Equation (3.11) is our final formula for the Feynman propagator. That this is the Feynman propagator follows from its derivation (with the representation of the time evolution in section 2 and [11]). We can also check directly through differentiation that K_t (equation (3.11)) satisfies the Schrödinger equation (the initial condition $K_0(x,y) = \delta(x-y)$ is obviously satisfied).

The expectation value (3.11) is finite under the assumptions (3.1)–(3.2), which for the holomorphic function (3.7) ($x_2 = 0$) read

$$|\operatorname{Im} V(x + y + \mathrm{i}y)| \leqslant A(x)(|y| + 1). \tag{3.12}$$

The assumption (3.12) coincides with that of Doss–Azencott [11, 12]. So the formula (3.11) applies at least to Doss potentials V [11]. We shall extend the method to potentials from $\mathcal{H}_n{}^{\mathcal{F}}$ in the next section.

Remark 1. The kernel $K_t(x,y)$ (3.11) is well defined also for some meromorphic potentials V if we exclude a set of points (x,y) of measure zero (we shall discuss such potentials in section 6).

Remark 2. It can be seen from equation (3.11) that $K_t(\lambda x, \lambda y)$ is well defined (i.e. the expectation value is finite) for a monomial V, bounded from below, of order $N = 4p$ or $N = 8p + 6$, where $p = 0, 1, \dots$ (it is easy to see that $K_t(x,y)$ itself is well defined for any polynomial, bounded from below, of order $N = 8p + 6$). So we could extend the validity of the formula (3.11) to some even polynomials V through an analytic continuation in space.

4. Definition of the Feynman kernel for potentials from $\mathcal{H}_n^{\mathcal{F}}$

Equation (3.11) as it stands is not directly applicable to potentials from $\mathcal{H}_n{}^{\mathcal{F}}$, because it is *not clear whether the expression under the expectation value is integrable for such potentials* (they do not satisfy the condition (3.2)). We proceed in the same way as we did for the wavefunction (2.8). We first regularize the potential $V \to V^R$ in the kernel $\mathcal{R}$ in equation (3.11). We obtain in this way the regularized kernels $\mathcal{R}_t{}^R$. We prove:

Theorem 4.1. Define

$$\mathcal{R}_t{}^R(x, y) = E\left[\exp\left\{\frac{g}{\lambda^2\hbar}\int_0^t \theta_R\left(\alpha\left(\frac{s}{t}\right)\right) V\left(x\left(1-\frac{s}{t}\right) + y\,\frac{s}{t} + \lambda\sigma\sqrt{t}\,\alpha\left(\frac{s}{t}\right)\right) \mathrm{d}s\right\}\right] \tag{4.1}$$

where $V \in \mathcal{H}_n{}^{\mathcal{F}}$. The limit $R \to \infty$ exists uniformly in t, x, y. $\mathcal{R}_t$ is an analytic function of g and $\lambda \neq 0$ if $\operatorname{Re}\lambda^2 \geqslant 0$. $K^0{}_t\mathcal{R}_t$ defines the Feynman propagator (3.11) for $V \in \mathcal{H}_n{}^{\mathcal{F}}$.

Proof. We expand the exponential in equation (4.1) in a power series in g. As in the proof of theorem 2.2, using the Lebesgue dominated convergence theorem we exchange the sum with the expectation value. We obtain

$$\mathcal{R}_t{}^R = \lim_{N\to\infty} \mathcal{R}_N(R) \tag{4.2}$$

where

$$\mathcal{R}_N(R) = \sum_{r=0}^{N} \frac{1}{r!}\left(\frac{g}{\lambda^2\hbar}\right)^r$$

$$\times\, E\left[\int_0^t \mathrm{d}s_1 \ldots \int_0^t \mathrm{d}s_r\, \mathrm{d}\nu_V(\gamma_1)\ldots \mathrm{d}\nu_V(\gamma_r)\, \exp\left(-\mathcal{B}_r{}^R + \mathrm{i}\sum_{k=1}^{r}\gamma_k p(s_k)\right)\right]$$

with

$$\mathcal{B}_r{}^R = \frac{t}{2R}\sum_{k=1}^{r}\alpha\left(\frac{s_k}{t}\right)^2 - \mathrm{i}\sigma\lambda\sqrt{t}\sum_{k=1}^{r}\gamma_k\alpha\left(\frac{s_k}{t}\right)$$

and

$$p(s) = x\left(1-\frac{s}{t}\right) + y\,\frac{s}{t}\,.$$

Then, owing to the assumption (2.2) we can apply the Fubini theorem

$$E\left[\int \mathrm{d}\nu_V(\gamma_1)\ldots\mathrm{d}\nu_V(\gamma_r)\, \exp\left(-\mathcal{B}_r{}^R\right)\right] = \int \mathrm{d}\nu_V(\gamma_1)\ldots\mathrm{d}\nu_V(\gamma_r)\, E\left[\exp\left(-\mathcal{B}_r{}^R\right)\right]\,.$$

Next, we show the existence of the limit

$$\lim_{R\to\infty} \mathcal{R}_N(R) \equiv \mathcal{R}_N\,. \tag{4.3}$$

This follows from the Lebesgue dominated convergence theorem, because the measure $|\nu|$ is finite and

$$|E[\exp(-\mathcal{B}_r{}^R)]| \leqslant 1 \tag{4.4}$$

by virtue of lemma 2.1. From the bound (4.4) it follows that

$$|\mathcal{R}_N(R) - \mathcal{R}_M(R)| \leqslant \sum_{r=M}^{N} \left(\frac{|g|t}{\hbar|\lambda|^2}\right)^r \frac{|\nu|^r}{r!}$$

where $|\nu|$ is the total variation of ν. Hence the convergence $N \to \infty$ of $\mathcal{R}_N(R)$ is uniform in R (as well as in x and t). Therefore we may exchange the limits $N \to \infty$ and $R \to \infty$

$$\begin{aligned}
\mathcal{R} &\equiv \lim_{R\to\infty} \mathcal{R}^R = \lim_{R\to\infty}\lim_{N\to\infty} \mathcal{R}_N(R) = \lim_{N\to\infty}\lim_{R\to\infty} \mathcal{R}_N(R) \\
&= \sum_r^\infty (\lambda^{-2})^r \hbar^{-r} \frac{g^r}{r!} E\left[\left\{\int V\left(p(\tau) + \lambda\sigma\sqrt{t}\,\alpha\left(\frac{\tau}{t}\right)\mathrm{d}\tau\right)\right\}^r\right] \\
&= \sum_r^\infty (\lambda^{-2})^r \hbar^{-r} \frac{g^r}{r!} \int_0^t \mathrm{d}\tau_1 \ldots \int_0^t \mathrm{d}\tau_r \int \mathrm{d}\nu_V(\gamma_1)\ldots\mathrm{d}\nu_V(\gamma_r) \\
&\quad \times \exp\left\{\mathrm{i}\sum_k \gamma_k p(\tau_k)\right\} \exp\left\{-\tfrac{1}{2}t\sigma^2\lambda^2 E\left[\left(\sum_k \gamma_k \alpha\left(\frac{\tau_k}{t}\right)\right)^2\right]\right\}.
\end{aligned} \tag{4.5}$$

Finally, we can prove that $K^0{}_t\mathcal{R}_t$ is the Feynman propagator, differentiating equation (4.5) term by term. In fact, after a computation of the expectation values in equation (4.5) we can convince ourselves that the expansion (4.5) coincides with the standard Dyson expansion of the Feynman propagator. □

5. Semiclassical approximation to the solution of the Schrödinger equation

The semiclassical expansion of the solution of the Schrödinger equation is discussed by Maslov [18] for arbitrarily large time. A simple rigourous version of this expansion for a small time has been proved by Truman [19]. If the initial wavefunction is analytic and an analytic potential is of the Doss class, then the semiclassical expansion for small time has been established by Azencott and Doss [12]. The expansion is based on the following lemma proved in [12].

Lemma 5.1. Assume that $F(b+\zeta f)$ is an analytic function of ζ. Then for any integrable F (depending on $b(s)$ with $0 \leqslant s \leqslant t$) and for any $\zeta \in \mathbb{C}$

$$E[F(b)] = E\left[F(b+\zeta f)\exp\left(-\tfrac{1}{2}\zeta^2\int_0^t \left(\frac{\mathrm{d}f}{\mathrm{d}s}\right)^2 \mathrm{d}s\right)\exp\left(-\zeta\int_0^t \frac{\mathrm{d}f}{\mathrm{d}s}\,\mathrm{d}b(s)\right)\right] \tag{5.1}$$

if f' is square integrable.

If $\zeta \in \mathbb{R}$ then equation (5.1) is the standard Cameron–Martin formula [17]. Then, because the r.h.s. is analytic in $\zeta \in \mathbb{C}$ and independent of ζ if $\zeta \in \mathbb{R}$ it follows that it does not depend on $\zeta \in \mathbb{C}$.

Applying equation (5.1) to $\psi_t{}^R$ (2.8) we obtain the following identity

$$\psi_t{}^R(x) = E\Bigg[\Omega_t\left(g\theta_R(b+\zeta f); V(x+\lambda\sigma b+\lambda\sigma\zeta f)\right)\theta_R(b+\zeta f)\psi(x+\lambda\sigma b_t+\zeta\lambda\sigma f)$$
$$\times \exp\left(-\tfrac{1}{2}\zeta^2\int_0^t\left(\frac{\mathrm{d}f}{\mathrm{d}s}\right)^2-\zeta\int_0^t\frac{\mathrm{d}f}{\mathrm{d}s}\,\mathrm{d}b\right)\Bigg]. \tag{5.2}$$

We can prove an analogue of theorem 2.2.

Theorem 5.1. Let $\psi_t{}^R$ be defined by equation (5.2), then the limit ψ_t of $\psi_t{}^R$ when $R\to\infty$ exists for arbitrary t, x and square integrable f'. ψ_t is the solution of the Schrödinger equation (2.4). It can be expressed by the absolutely convergent series

$$\psi_t \equiv \lim_{N\to\infty}\psi_t{}^N = \lim_{N\to\infty}\exp\left(-\tfrac{1}{2}\zeta^2\int_0^t\left(\frac{\mathrm{d}f}{\mathrm{d}\tau}\right)^2\right)\sum_{n=0}^{N}(\lambda^{-2})^n\hbar^{-n}\frac{g^n}{n!}$$
$$\times E\Bigg[\exp\left(-\zeta\int_0^t\frac{\mathrm{d}f}{\mathrm{d}\tau}\mathrm{d}b\right)\left(\int_0^t V(x+\lambda\sigma b_\tau+\zeta\lambda\sigma f_\tau)\right)^n$$
$$\times\ \psi(x+\lambda\sigma b_t+\zeta\lambda\sigma f_t)\Bigg]. \tag{5.3}$$

Proof. We repeat the arguments of the proof of theorem 2.2 applied to equation (5.2). □

We now choose $\zeta=\lambda^{-1}$. Let us denote

$$\xi_s(x) = x+\sigma f(s). \tag{5.4}$$

Assume that ψ, being a Fourier transform of a bounded measure, has at the same time the semiclassical form

$$\psi = \int \exp(\mathrm{i}x\gamma)\,\mathrm{d}\nu_\psi = \exp\left(\frac{\mathrm{i}W}{\hbar}\right)\phi \tag{5.5}$$

where

$$\phi(x) = \int \exp(\mathrm{i}x\gamma)\,\mathrm{d}\nu_\phi(\gamma).$$

We choose f in such a way that ξ is the solution of the boundary value problem (for a sufficiently small time there exists the unique solution of this problem [21])

$$\begin{aligned} &m\frac{\mathrm{d}^2\xi}{\mathrm{d}s^2} = -\nabla V(\xi)\\ &\xi_s|_{s=0}=x \qquad \text{and} \qquad m\left.\frac{\mathrm{d}\xi}{\mathrm{d}s}\right|_{s=t} = \nabla W(\xi_t). \end{aligned} \tag{5.6}$$

Next let us denote the classical action corresponding to the trajectory (5.6) by S

$$S_t(\xi) = W(\xi_t) + \tfrac{1}{2}m\int_0^t \left(\frac{d\xi}{ds}\right)^2 ds - \int_0^t gV(\xi_s)\, ds.$$

$S_t(\xi)$ is the solution of the classical Hamilton–Jacobi equation

$$\partial_t S + \frac{(\nabla S)^2}{2m} + V = 0 \tag{5.7}$$

with the initial condition $S_t|_{t=0} = W$.

The virtue of the boundary condition (5.6) is that the terms linear in $\boldsymbol{b}$ cancel in the formal limit $R \to \infty$. In this limit $\psi_t(x)$ depends on

$$V_2(\xi + \lambda\sigma \boldsymbol{b}) \equiv V(\xi + \lambda\sigma \boldsymbol{b}) - V(\xi) - \nabla V(\xi)\lambda\sigma \boldsymbol{b}$$

and

$$W_2(\xi + \lambda\sigma \boldsymbol{b}) \equiv W(\xi + \lambda\sigma \boldsymbol{b}) - W(\xi) - \nabla W(\xi)\lambda\sigma \boldsymbol{b} \tag{5.8}$$

which are of order $\hbar$. Hence,

$$\psi_t(x) \sim \exp\left(\frac{\mathrm{i}S_t(\xi)}{\hbar}\right)\phi(\xi_t). \tag{5.9}$$

We are going to give equation (5.9) a rigourous meaning for $V \in \mathcal{H}_n{}^{\mathcal{F}}$ (for Doss potentials the semiclassical formula has been proved in [12]).

First, we must restrict the growth of Im W on the subset $\mathbb{G}_x \subset \mathbb{R}^{2n}$ (see the definitions (3.1) and (3.7)). Assume

$$-\operatorname{Im} W_2(x + y + \mathrm{i}y) \leqslant K(x)|y|^2 \tag{5.10}$$

with a certain non-negative continuous function K. We need the bound (5.10) in order to prove integrability.

Lemma 5.2. Let $\mathcal{M}$ be an $n \times n$ matrix, then

$$E[\exp(\tfrac{1}{2}\boldsymbol{b}_t\mathcal{M}\boldsymbol{b}_t)] < \infty$$

if $|\mathcal{M}| < 1/t$.

Proof. The statement follows from the formula for the expectation value

$$\int dy\,(2\pi t)^{-n/2}\exp\left(-\frac{|y|^2}{2t} + \tfrac{1}{2}y\mathcal{M}y\right). \qquad \square$$

Theorem 5.3. Assume the bound (5.10), then for each x and t sufficiently small

$$\lim_{\hbar\to 0}\psi_t(x)\exp\left(-\frac{\mathrm{i}}{\hbar}S_t(\xi)\right) \equiv \Phi(t, x) \tag{5.11}$$

Φ can be expressed explicitly by a convergent series

$$\Phi(t,x)=\lim_{\hbar\to0}\sum_{n}^{\infty}(\lambda^{-2})^{n}\frac{g^{n}}{n!}E\left[\left(\hbar^{-1}\int V_{2}(\xi_{\tau}+\lambda\sigma b_{\tau})\,d\tau\right)^{n}\exp\left(\frac{i}{\hbar}W_{2}\right)\phi(\xi_{t}+\lambda\sigma b_{\tau})\right]$$

$$=\phi(\xi_{t})\sum_{n}^{\infty}(\lambda^{-2})^{n}(-2)^{-n}\frac{g^{n}}{n!}\int_{0}^{t}d\tau_{1}\ldots\int_{0}^{t}d\tau_{n}\int d\nu_{V}(\gamma_{1})\ldots d\nu_{V}(\gamma_{n})$$

$$\times\exp\left(i\sum_{k}\gamma_{k}\xi_{\tau_{k}}\right)E\left[\prod_{k}(b_{\tau_{k}}\gamma_{k})^{2}\exp\left(-\frac{1}{2m}b_{t}W''b_{t}\right)\right]$$

$$=E\left[\exp\left\{\frac{1}{2m}\int_{0}^{t}b(s)V''(\xi_{s})b(s)\,ds-\frac{1}{2m}b_{t}W''b_{t}\right\}\right]\phi(\xi_{t}) \qquad (5.12)$$

where V'' and W'' denote the second-order Frechet derivatives.

Proof. We use the decomposition (5.8) in equation (5.3) (with $\lambda^2 = i$). We rewrite the absolutely convergent series (5.3) in such a way that the regularity in $\hbar$ is visible. So, we subtract $W(\xi_t)+\nabla W(\xi_t)\lambda\sigma b_t$ from $W(\xi_t+\lambda\sigma b_t)$ (equation (5.8)) and $V(\xi_\tau)+\nabla V(\xi_\tau)\lambda\sigma b_\tau$ from $V(\xi_\tau+\lambda\sigma b_\tau)$ where from the classical equations of motion ∇V can be expressed by f'' (equations (5.6)). Expanding in the coupling constant g we rewrite the series (5.3) in the form

$$\exp\left(-\frac{i}{\hbar}S\right)\psi_{t}\equiv\exp\left(-\frac{i}{\hbar}S\right)\lim_{N\to\infty}\psi_{t}{}^{N}$$

$$=\lim_{N\to\infty}\sum_{n=0}^{N}i^{-n}g^{n}E\left[\lim_{K\to\infty}\sum_{k=0}^{K}\left(\frac{i}{\hbar}\int_{0}^{t}V(\xi_{\tau})\,d\tau-\frac{1}{\lambda}\int_{0}^{t}\frac{d^{2}f}{d\tau^{2}}b\right)^{k}\right.$$

$$\times\sum_{m=0}^{m=n}\frac{1}{m!(n-m)!}\int_{0}^{t}\cdots\int_{0}^{t}(i\hbar)^{-m}(V_{2})^{m}(i\hbar)^{-n+m}(V+\lambda\sigma b\nabla V)^{n-m}$$

$$\left.\times\exp\left(\frac{iW_{2}}{\hbar}\right)\phi(\xi_{t}+\lambda\sigma b_{t})\right]$$

$$=\lim_{N\to\infty}\lim_{K\to\infty}F_{N,K}=\lim_{N\to\infty}F_{N,N}\,. \qquad (5.13)$$

In equation (5.13) the potentials V and V_2 depend on paths, which are functions of time τ. The multiple integral denotes an integral over all these τ. Then

$$F_{N,N}=\sum_{n=0}^{N}i^{-n}\frac{g^{n}}{n!}E\left[\left(\int_{0}^{t}\hbar^{-1}V_{2}\right)^{n}\exp\left(\frac{i}{\hbar}W_{2}\right)\phi(\xi+\lambda\sigma b)\right]. \qquad (5.14)$$

In the expansion (5.13) $\nabla W\lambda\sigma b$ cancelled as a result of the integration by parts in $\int_0^t f'\,db$ in equation (5.3) and the boundary conditions (5.6). Owing to our assumption (5.10) and lemma 5.2, each term in the series (5.13) is finite for sufficiently small t. The subsequent transformations in the formula (5.13) are allowed, because the series $\lim_{K\to\infty} F_{N,K}$ (fixed N) is absolutely convergent. So, owing to the Lebesgue dominated convergence we were able to exchange the limit $K \to \infty$ and the expectation value. Again, by the absolute convergence we were allowed to take the limit of the double sequence $F_{N,K}$ over the diagonal. When $N = K$ in equation (5.13) the terms in the sum independent of b then cancel identically, whereas the terms linear in b cancel each other owing to the equations of motion (5.6). Next, we prove that if t is small enough, then the convergence for $N \to \infty$ of the series (5.14) is uniform in $\hbar$. First, note that the series (5.14) is absolutely convergent for t small enough. In fact, using the Taylor formula with a remainder

$$V_2(\xi + \lambda\sigma b) = \int_0^1 \mathrm{d}s \, \frac{\mathrm{d}^2}{\mathrm{d}s^2} \, V(\xi + \lambda\sigma s b)$$

we can express the nth order term in equation (5.14) in the form

$$\begin{aligned} \frac{1}{n!} \exp\left(-\frac{\mathrm{i}W(\xi_t)}{\hbar}\right) E\bigg[\int_0^t \mathrm{d}\tau_1 \ldots \int_0^t \mathrm{d}\tau_n \int_0^1 \mathrm{d}s_1 \frac{\mathrm{d}^2}{\mathrm{d}s_1^2} \cdots \int_0^1 \mathrm{d}s_n \frac{\mathrm{d}^2}{\mathrm{d}s_n^2} \\ \times \int \mathrm{d}\nu_V(\gamma_1) \ldots \mathrm{d}\nu_V(\gamma_n) \, \mathrm{d}\nu_\psi(\beta) \exp\left(-\frac{\mathrm{i}\sigma\lambda b \nabla W(\xi_t)}{\hbar}\right) \\ \times \exp\left(\mathrm{i}\lambda\sigma \sum_{p=1}^n \gamma_p \xi(\tau_p) + \mathrm{i}\lambda\sigma\beta\xi(t) + \mathrm{i}\sigma\lambda\beta b(t) + \mathrm{i}\lambda\sigma \sum_{m=1}^n \gamma_m b(\tau_m) s_m\right)\bigg] . \end{aligned} \tag{5.15}$$

We can compute the expectation value in equation (5.15) explicitly (the integral is Gaussian). After the computation of the expectation value we obtain a Gaussian function of s_k. An estimate of the number of terms resulting from a differentiation over s can be made analytically using a representation

$$\frac{\partial}{\partial s_k} \exp(-\tfrac{1}{2}\mathrm{i}sMs) = \det(2\pi\mathrm{i}M)^{-1/2} \int \mathrm{d}u \, \exp(\tfrac{1}{2}\mathrm{i}uM^{-1}u) \, \frac{\partial}{\partial s_k} \exp(\mathrm{i}us) \tag{5.16}$$

true (with a proper interpretation of the square root of the determinant in equation (5.16)) for any invertible real matrix M. Now, it can be seen from equation (5.16) that the expectation value (5.15) is bounded by $n!$, because the $2n$ differentiations in equation (5.16) give a factor $n!$ times a bounded function (from the well known properties of the Gaussian integral). We could also obtain this bound directly without the use of the representation (5.16) if we were to note that the nth term in the series (5.15) is equal to a Wiener integral of the form $\sim E[b^{2n}]$ times a phase factor. We conclude that the number of terms generated by the differentiation over s is bounded by $n!$. Hence, because the expectation value (5.15) (before the differentiation over s) is a phase factor (owing to lemma 2.1) we obtain that the expression (5.15) is bounded by $K^n t^n$ for a certain K. Therefore the series (5.14) is absolutely convergent uniformly in $\hbar$ for sufficiently small time. Taking the limit $\hbar \to 0$ term by term we arrive at the formula (5.12). □

Remark 1. When we apply the Schwarz inequality to the RHS of equation (5.12) and use lemma 2.3 and lemma 5.2, we then obtain a necessary condition on t (which we believe is close to optimal)

$$t < \tfrac{1}{2}\pi(\sup|V''(\xi)|)^{-1} + \tfrac{1}{2}(\sup|W''(\xi)|)^{-1} \tag{5.17}$$

(the supremum is taken on the trajectory $\xi(\tau)$ with $0 \leqslant \tau \leqslant t$).

Remark 2. We could consider meromorphic initial conditions W. In such a case the bound (5.10) can be satisfied except for a discrete set of points x. When the bound (5.10) holds, then the expectation values (5.3) and (5.12) are finite. $\psi_t(x)$ (5.3) is the solution of the Schrödinger equation except for a neighbourhood of a discrete set of points, and equation (5.12) determines its semiclassical limit.

6. Semiclassical expansion of the Feynman propagator

Let us consider first a simple case of a meromorphic potential V of the form

$$V(x) = Q(x)P(x)^{-1} \tag{6.1}$$

where Q and P are polynomials and their degrees satisfy the inequality $\deg(Q) \leqslant \deg(P) + 2$. Then $V(z)$ is quadratically bounded for $z \in \mathbb{C}^n$, except for a discrete set of points corresponding to zeros of P. The potential in equation (3.11) can be singular at a discrete set of values of $\alpha(s)$. With such singularities the expectation value in equation (3.11) still can be finite. However, in order to avoid eventual difficulties we restrict ourselves here to x and y from a set

$$\mathbb{R}_0{}^{2n} \equiv \left\{(x, y) | P\left(x\left(1 - \frac{s}{t}\right) + y\frac{s}{t} + \lambda\sigma\alpha\left(\frac{s}{t}\right)\right) \neq 0\right\} \subset \mathbb{R}^n \times \mathbb{R}^n.$$

It is easy to see that the condition that $(x, y) \notin \mathbb{R}_0{}^{2n}$ reads

$$x\left(1 - \frac{s}{t}\right) + y\frac{s}{t} = \operatorname{Re} z_0 - \operatorname{Im} z_0 \tag{6.2}$$

for a certain $0 \leqslant s \leqslant t$ where z_0 is a (complex) zero of P, ie $(x, y) \notin \mathbb{R}_0{}^{2n}$ if $\operatorname{Re} z_0 - \operatorname{Im} z_0$ lies on the line joining x and y.

In order to derive a semiclassical expansion we make a shift of variables in equation (3.11) similarly as in equation (5.2)

$$\alpha \to \alpha + \zeta f. \tag{6.3}$$

After the shift (6.3) we obtain (on setting the coupling constant g of (2.3) equal to 1)

$$\begin{aligned} \mathcal{R}_t(x, y) = \mathcal{R}_t{}^{\zeta}(x, y) = \exp\left(-\tfrac{1}{2}\zeta^2 \int_0^t \left(\frac{\mathrm{d}f}{\mathrm{d}\tau}\right)^2 \mathrm{d}\tau\right) E\left[\exp\left\{-\zeta \int_0^t \frac{\mathrm{d}f(\tau)}{\mathrm{d}\tau}\,\mathrm{d}\alpha(\tau)\right.\right. \\ \left.\left. - \frac{\mathrm{i}}{\hbar}\int_0^t V\left(x\left(1 - \frac{s}{t}\right) + y\frac{s}{t} + \lambda\sigma\sqrt{t}\left(\alpha\left(\frac{s}{t}\right) + \zeta f\left(\frac{s}{t}\right)\right)\right)\mathrm{d}s\right\}\right]. \end{aligned} \tag{6.4}$$

Equation (6.4) holds true for an arbitrary f defined on the interval $[0, 1]$ whose derivative is square integrable and $f(1) = f(0) = 0$ (f must satisfy the same boundary conditions as the Brownian bridge does).

The potential V, as well as the expectation value (6.4), are analytic in ζ. In such a case, from the fact that $\mathcal{R}_t{}^{\zeta}(x, y)$ in equation (6.4) does not depend on ζ for ζ real, we can conclude that it does not depend on ζ if ζ is allowed to be complex. We choose now $\zeta = \lambda^{-1}$ in equation (6.4). Let us denote

$$q(s; x, y) = x\left(1 - \frac{s}{t}\right) + y\frac{s}{t} + \sigma\sqrt{t} f\left(\frac{s}{t}\right) \tag{6.5}$$

$q \in \mathbb{R}^n$ is a curve joining x to y. We choose it as a solution of the Newton equation

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} = -\nabla V(q) \tag{6.6}$$

with the boundary condition $q(0) = x$ and $q(t) = y$ (under our assumptions on V there exists a unique solution of equation (6.6), at least for small time). We assume that $q(s) \neq \mathrm{Re}\, z_0 - \mathrm{Im}\, z_0$ for any s. This condition is not so easy to verify and looks rather awkward from the point of view of classical dynamics. It could be justified by a requirement that the classical (real) dynamics has an extension to complex dynamics.

Let us introduce the classical action

$$S_t(q) = \tfrac{1}{2}m \int_0^t \left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 \mathrm{d}s - \int_0^t V(q(s))\,\mathrm{d}s. \tag{6.7}$$

Then the formula for $\mathcal{R}$ (6.4) takes the form

$$\begin{aligned}\mathcal{R}_t(x, y) &= \exp\left(\frac{\mathrm{i}S_t(q)}{\hbar}\right) E\left[\exp\left\{-\frac{\mathrm{i}}{\hbar}\int_0^t V_2\left(q(s) + \lambda\sigma\sqrt{t}\alpha\left(\frac{s}{t}\right)\right)\mathrm{d}s\right\}\right]\exp\left(\frac{(x-y)^2}{2it\sigma^2}\right)\\ &\equiv \exp\left(\frac{\mathrm{i}S_t(q)}{\hbar}\right)\exp\left(\frac{(x-y)^2}{2it\sigma^2}\right)\mathcal{R}^{\hbar}\end{aligned} \tag{6.8}$$

where

$$V_2(q + \lambda\sigma\sqrt{t}\,\alpha) \equiv V(q + \lambda\sigma\sqrt{t}\,\alpha) - V(q) - \lambda\sigma\sqrt{t}\,\alpha\nabla V(q). \tag{6.9}$$

Theorem 6.1. Let V be the rational function (6.1). Assume that $q(s) \neq \mathrm{Re}\, z_0 - \mathrm{Im}\, z_0$ for any s and any zero z_0 of P. Then for sufficiently small t

$$\lim_{\hbar\to 0} K_t(x, y)(2\pi it\sigma^2)^{n/2}\exp\left(-\frac{\mathrm{i}}{\hbar}S_t(q)\right) = E\left[\exp\left\{\frac{t}{2m}\int_0^t \alpha\left(\frac{s}{t}\right)V''(q(s))\alpha\left(\frac{s}{t}\right)\mathrm{d}s\right\}\right] \tag{6.10}$$

where V'' denotes the second-order Frechet derivative.

Moreover, the expansion of $\mathcal{R}^{\hbar}$ in σ is asymptotic.

Proof. In order to prove that the expansion of $\mathcal{R}^{\hbar}$ (equation (6.8)) in σ is asymptotic it is sufficient to show that $\mathcal{R}^{\hbar}$ and all its derivatives over σ are bounded uniformly in $0 \leqslant \sigma \leqslant \epsilon$ for a certain ϵ. First, if V is a rational function (6.1) then V_2 (6.9) is also a rational function of the same type (quadratically bounded) with the same discrete set of singularities. Now for Gaussian integrals we have [16]

$$E\left[\mathcal{P}(\alpha)\exp\left(\tfrac{1}{2}t\int_0^1 \omega(\tau)^2\alpha(\tau)^2\,\mathrm{d}\tau\right)\right] < \infty \tag{6.11}$$

if $t^2 < \pi^2(\sup_\tau |\omega(\tau)|^2)^{-1}$ for any polynomially bounded function $\mathcal{P}$ (this is a sufficient condition for a finiteness of the integral (6.11), not a necessary one). From equation (6.11) it follows that $\mathcal{R}^{\hbar}$ is integrable uniformaly in σ, because V_2 is quadratically bounded. Applying the Lebesgue dominated convergence, we obtain the explicit formula (6.10) for the limit $\sigma \to 0$. It follows also from equation (6.11) that the integral (6.10) is finite if

$$t^2 \sup_s V''(q(s)) < m\pi^2. \tag{6.12}$$

It remains to show that the derivatives are also uniformly bounded. We have

$$\left|\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\mathcal{R}^{\hbar}{}_{t}\right)\right| \leqslant E\left[\left|\frac{\mathrm{d}}{\mathrm{d}\sigma}\frac{1}{m\sigma^2}\int_0^t V_2\left(q(\tau)+\lambda\sigma\sqrt{t}\,\alpha\left(\frac{\tau}{t}\right)\right)\mathrm{d}\tau\right|\right.$$
$$\left.\times\exp\left\{\frac{1}{m\sigma^2}\int_0^t \operatorname{Im} V_2\left(q(s)+\lambda\sigma\sqrt{t}\,\alpha\left(\frac{s}{t}\right)\right)\right\}\right]. \tag{6.13}$$

For the rational potential (6.1) $\mathrm{d}/\mathrm{d}\sigma\,\sigma^{-2}V_2$ in equation (6.13) is bounded uniformly in σ and is polynomially bounded in α. $\operatorname{Im} V_2$ in the exponential in equation (6.13) is bounded uniformly in σ by a quadradtic form in α of the type (6.11). Hence, from the formula (6.11) it follows that the RHS of equation (6.13) is bounded uniformly in σ if time is small enough. It is clear from the argument applied to the first derivative that we can continue differentiation in σ and that the derivatives will be bounded uniformly in σ. Hence the semiclassical expansion is asymptotic for sufficiently small time. □

We would like to prove the semiclassical formula also for potentials of the form of a Fourier transform of a bounded measure. We obtain (here restoring the coupling g):

Theorem 6.2. Assume $V \in \mathcal{H}_n{}^{\mathcal{F}}$, then for sufficiently small t

$$\lim_{\hbar\to 0} K_t(x,y)(2\pi\sigma^2 \mathrm{i}t)^{n/2}\exp\left(-\frac{\mathrm{i}S_t(q)}{\hbar}\right) = E\left[\exp\left(\frac{t}{2m}g\int_0^t \alpha\left(\frac{s}{t}\right)V''(q(s))\alpha\left(\frac{s}{t}\right)\mathrm{d}s\right)\right]. \tag{6.14}$$

Proof. After the shift (6.3), equation (4.1) takes the form

$$\mathcal{R}_t{}^R(x,y) = E\left[\exp\left(\tfrac{1}{2}\mathrm{i}\int_0^t\left(\frac{\mathrm{d}f}{\mathrm{d}s}\right)^2 - \lambda^{-1}\int_0^t\frac{\mathrm{d}f}{\mathrm{d}s}\,\mathrm{d}\alpha\right)\right. \tag{6.15}$$
$$\left.\times\exp\left\{\frac{g}{\lambda^2\hbar}\int_0^t\theta_R\left(\alpha\left(\frac{s}{t}\right)+\lambda^{-1}f\left(\frac{s}{t}\right)\right)V\left(q\left(\frac{s}{t}\right)+\lambda\sigma\sqrt{t}\,\alpha\left(\frac{s}{t}\right)\right)\mathrm{d}s\right\}\right].$$

We can prove, as in theorem 4.1, that the limit $R\to\infty$ of $\mathcal{R}^R$ exists for arbitary f (with a square integrable derivative). It can be expressed by the absolutely convergent series

$$\mathcal{R} \equiv \lim_{N\to\infty}\mathcal{R}_N = \lim_{N\to\infty}\sum_{n=0}^N(\lambda^{-2})^n\frac{g^n}{n!} \tag{6.16}$$
$$\times E\left[\left\{\int\hbar^{-1}V\left(q(\tau)+\lambda\sigma\sqrt{t}\,\alpha\left(\frac{\tau}{t}\right)\right)\mathrm{d}\tau\right\}^n\right.$$
$$\left.\times\exp\left(\tfrac{1}{2}\mathrm{i}\int_0^t\left(\frac{\mathrm{d}f}{\mathrm{d}s}\right)^2-\lambda^{-1}\int_0^t\frac{\mathrm{d}f}{\mathrm{d}s}\,\mathrm{d}\alpha\right)\right].$$

As in the proof of theorem 5.3, we write

$$V(q+\sqrt{t}\,\lambda\sigma\alpha) = V(q)+\nabla V\lambda\sigma\sqrt{t}\,\alpha+V_2\,.$$

With this decomposition, if t is small enough

$$\exp\left(-\frac{\mathrm{i}}{\hbar}S\right)\mathcal{R}_t \equiv \exp\left(-\frac{\mathrm{i}}{\hbar}S\right)\lim_{N\to\infty}\mathcal{R}_t{}^N$$

$$= \lim_{N\to\infty}\sum_{n=0}^{N}\mathrm{i}^{-n}g^n E\left[\lim_{K\to\infty}\sum_{k=0}^{K}\left(\frac{\mathrm{i}}{\hbar}g\int_0^t V(q_\tau)\,\mathrm{d}\tau - \frac{1}{\lambda}\int_0^t \frac{\mathrm{d}^2 f}{\mathrm{d}\tau^2}\sqrt{t}\,\alpha\right)^k\right.$$

$$\left.\times\sum_{m=0}^{m=n}\frac{1}{m!(n-m)!}\int_0^t\ldots\int_0^t(\mathrm{i}\hbar)^{-m}(V_2)^m(\mathrm{i}\hbar)^{-n+m}(V+\lambda\sigma\sqrt{t}\,\alpha\nabla V)^{n-m}\right]$$

$$\equiv \lim_{N\to\infty}\lim_{K\to\infty}F_{N,K} = \lim_{N\to\infty}F_{N,N} \tag{6.17}$$

where the meaning of the multiple integral is the same as in equation (5.13) and

$$F_{N,N} = \sum_{n=0}^{N}\mathrm{i}^{-n}\frac{g^n}{n!}E\left[\left(\int_0^t \hbar^{-1}V_2\right)^n\right]. \tag{6.18}$$

The formula (6.17) holds true because the series $\lim_{K\to\infty}F_{N,K}$ (fixed N) is absolutely convergent. So, owing to the Lebesgue dominated convergence we were able to exchange the limit $K\to\infty$ and the expectation value. Again, by the absolute convergence we were allowed to take the limit of the double sequence $F_{N,K}$ over the diagonal, then the terms linear in α cancel each other owing to the equations of motion (6.6). Next we show that the series (6.18) is absolutely convergent uniformly in $\hbar$ for t small enough. In fact, using the Taylor formula with a remainder

$$V_2(q+\lambda\sigma\sqrt{t}\,\alpha) = \int_0^1 \mathrm{d}u\,\frac{\mathrm{d}^2}{\mathrm{d}u^2}V(q+\lambda\sigma u\sqrt{t}\,\alpha)$$

we can write the nth order term in equation (6.18) in the form

$$\frac{1}{n!}E\left[\int_0^t \mathrm{d}\tau_1\ldots\int_0^t \mathrm{d}\tau_n\int_0^1 \mathrm{d}u_1\frac{\mathrm{d}^2}{\mathrm{d}u_1^2}\ldots\int_0^1 \mathrm{d}u_n\frac{\mathrm{d}^2}{\mathrm{d}u_n^2}\int \mathrm{d}\nu_V(\gamma_1)\ldots\mathrm{d}\nu_V(\gamma_n)\right.$$

$$\left.\times\exp\left(\mathrm{i}\lambda\sigma\sum_{p=1}^{n}\gamma_p q(\tau_p) + \mathrm{i}\lambda\sigma\sqrt{t}\sum_{r=1}^{n}\gamma_r\alpha\left(\frac{\tau_r}{t}\right)u_r\right)\right]. \tag{6.19}$$

We can compute the expectation value in equation (6.19) explicitly. After the computation of the expectation value we differentiate over u. The estimate of the number of terms can be obtained in the same way as in the proof of theorem 5.3. We can conclude that the number of terms generated by differentiation is bounded by $n!$. Hence, because the expectation value (6.19) (before the differentiation over u) is a phase factor, we obtain that equation (6.19) is bounded by $K^n t^n$ for a certain K. So the series is absolutely convergent uniformly in $\hbar$. Taking the limit $\hbar\to 0$ term by term we arrive at the formula (6.14). □

If $\deg P + 1 \geqslant \deg Q$ *then the formula (6.8) is well defined for arbitrarily large* t. However, the remaining non-classical terms could give a large contribution for small σ. In order to get some feeling for what can happen, let us apply the Jensen inequality in the form

$$E\left[\int_0^t \frac{\mathrm{d}s}{t}\exp\left(A(s)\right)\right] \geqslant E\left[\exp\left(\int_0^t \frac{\mathrm{d}s}{t} A(s)\right)\right] \tag{6.20}$$

true for any real function A. Applying the Jensen inequality (6.20) and the Fubini theorem we can obtain the estimate

$$\begin{aligned} |\mathcal{R}^{\hbar}{}_t(x,y)| &\leqslant \int_0^t \frac{\mathrm{d}s}{t} E\left[\exp\left\{\frac{t}{\hbar}\operatorname{Im} V_2\left(q(s)+\lambda\sigma\sqrt{t}\,\alpha\left(\frac{s}{t}\right)\right)\right\}\right] \\ &= \hbar^{-n/2}\int_0^t \frac{\mathrm{d}s}{t}\int \mathrm{d}z\,(2\pi s)^{-n/2}(t-s)^{n/2}\exp\left(-\frac{(t-s)z^2}{2s\hbar}\right) \\ &\quad\times \exp\left\{\frac{t}{\hbar}\operatorname{Im} V_2\left(q(s)+\lambda\sqrt{\frac{t}{m}}\left(1-\frac{s}{t}\right)z\right)\right\}. \end{aligned} \tag{6.21}$$

In order to compute the expectation value (6.21) we have used the representation of the Brownian bridge in terms of the Brownian motion

$$\alpha(\tau) = (1-\tau)b\left(\frac{\tau}{1-\tau}\right)$$

(see [16]).

If $\operatorname{Im} V_2(q+z) \sim z$ for large z, then through the saddle point method we obtain, in general, for the RHS of equation (6.21) the behaviour $\exp(K/\hbar)$, where the constant K can be positive. If this is the true behaviour of the functional integral (6.8) then we have a profound departure from the semiclassical approximation. With the formula (6.8), which also holds true for non-integrable systems and time-dependent potentials, we have a method of studying the behaviour for small $\hbar$ and large t. An investigation of this regime is important for an understanding of the relation between classical and quantum mechanics. A detailed estimate of the behaviour of $\mathcal{R}_t$ for the whole range of t and $\hbar$ would be useful for applications of semiclassical methods. Apparently, the estimate $\exp(K/\hbar)$ is too pessimistic (a blow-up of the semiclassical approximation; there are some indications that the semiclassical behaviour of chaotic systems can continue for a large time [20]). First of all, in deriving the bound (6.21) we have taken the absolute value under the integral sign, neglecting the oscillatory terms in equation (6.8) which are crucial for the sign of K (at least, this is so for finite-dimensional integrals). The Jensen inequality applied in equation (6.21) may also be too rough. In addition, the condition (6.12) (resulting from the requirement (6.11)) is also too strong. It is not satisfied for large t on the RHS of equation (6.10). The RHS of equation (6.10) is equal to the van Vleck determinant. As is well known from the study of the van Vleck determinant, it becomes infinite at a certain time t_0 (owing to a focal point), but for $t > t_0$ it varies continuously until the next focal point.

The formula (6.8) gives a sound starting point for an investigation of the large t and small $\hbar$ behaviour. Let us mention here a simple application to a potential $V(x)$ of meromorphic type (6.1) decreasing to zero when $|x| \to \infty$. In such a case the saddle-point method applied to the integral on the RHS of equation (6.21) can give an estimate uniform in $\hbar$ and t

confirming the leading term of the semiclassical expansion (but not necessarily the van Vleck formula (6.10)). For rigourous estimates of this case as well as the more interesting cases with classical turning points, we need a detailed study of the saddle point method for the functional integral.

If the time is arbitrarily large, then the saddle points (5.6) and (6.6) may be absent, or there may be many saddle points [21]. In such a case the semiclassical estimates become much more involved.

Acknowledgment

The author would like to thank the anonymous referees for remarks which helped to improve this paper.

References

[1] Feynamn R P 1948 *Rev. Mod. Phys.* **20** 367
[2] Gelfand I M and Yaglom A M 1960 *J. Math. Phys.* **1** 48
[3] Cameron R H 1960 *J. Math. and Phys.* **39** 126
[4] Ito K 1967 *Proc. 5th Berkeley Symposium* ed J Neyman (Berkeley, CA: University of California Press)
[5] Albeverio S and Hoegh-Krohn R 1976 *Mathematical Theory of Feynman Path Integrals* (Heidelberg: Springer)
[6] Elworthy K D and Truman A 1984 *Ann. Inst. H. Poincaré* A **41** 115
[7] Ito K 1961 *Proc. 4th Berkeley Symposium* ed J Neyman (Berkeley, CA: University of California Press)
[8] Nelson E 1964 *J. Math. Phys.* **5** 332
[9] Feldman J 1963 *Trans. Am. Math. Soc.* **108** 251
[10] Streit L and Hida T 1983 *Stoch. Proc. Appl.* **16** 55
de Faria M, Potthoff J and Streit L 1991 *J. Math. Phys.* **32** 2123
[11] Doss H 1980 *Commun. Math. Phys.* **73** 247
[12] Azencott R and Doss H 1985 (*Lecture Notes in Mathematics 1109*) (Heidelberg: Springer)
[13] Albeverio S and Hoegh-Krohn R 1977 *Invent. Math.* **40** 59
[14] Skorohod A V 1974 *Integration in Hilbert Spaces* (Heidelberg: Springer)
[15] Carlen E and Kree P 1991 *Ann. Prob.* **19** 354
[16] Simon B 1979 *Functional Integration in Quantum Physics* (New York: Academic)
[17] Ikeda N and Watanabe S 1981 *Stochastic Differential Equations and Diffusion Processes* (Amsterdam: North-Holland)
[18] Maslov V P 1972 *Théorie des Perturbations et Methodes Asymptotiques* (Paris: Dunod)
[19] Truman A 1977 *J. Math. Phys.* **18** 2308
[20] Sepulveda M A, Tomsovic S and Heller E J 1992 *Phys. Rev. Lett.* **69** 42
[21] Bailey P B, Shampine L F and Waltman P E 1968 *Nonlinear Two-Point Boundary Value Problems* (New York: Academic)